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MA110 - Engineering Mathematics-1
Problem Sheet - 3

Partial Derivatives

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1. Verify the mixed derivative theorem for the following functions at the given point.

   (a) $f(x, y) = \sin xy$ at $(0, 0)$.

   (b) $f(x, y) = \frac{x+y}{x^2+y^2}$ at $(1, 1)$.

2. For the function f defined by

$$f(x, y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases} .$$

   Verify whether $f_{xy}(0, 0) = f_{yx}(0, 0)$.

3. Show that $f(x, y)$ is continuous and differentiable by definition.

   (a) $f(x, y) = xy$

   (b) $f(x, y) = x^2 + y^3$

4. Show that

$$f(x, y) = |x|$$

   is not differentiable at $(0, 0.)$

5. Show that

$$f(x, y) = |x|(1 + y)$$

   is not differentiable at $(0, 0.)$

6. Show that

$$f(x, y) = \sqrt{|xy|}$$

   is not differentiable at $(0, 0.)$

7. Use the limit definition to find

$$\frac{\partial f}{\partial x}(1, 2)$$

   of $f(x, y) = 1 - x + y - 3x^2y$.

8. Find $\frac{\partial z}{\partial x}(1, 1)$, if $xy + z^3x - 2yz = 0$.

9. Find the second-order partial derivatives for each of the functions defined below:

(a) $f(x,y) = \tan^{-1}\frac{x}{y}$

(b) $f(x,y) = e^x(x\cos y - y\sin x)$

(c) $f(x,y) = \log(x^3 + y^3 xy)^{\frac{1}{3}}$

(d) $f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$

(e) $f(x,y) = \log\frac{xy}{x^2+y^2}$.

10. Verify whether $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for the following functions:

(a) $f(x,y) = (x+y)\tan xy$

(b) $f(x,y) = \log xy$

(c) $f(x,y) = (x+y)\log xy$

(d) $f(x,y) = \sqrt{xy}$

(e) $f(x,y) = \frac{x+y}{x^2+y^2}$

(f) $f(x,y) = x^2\sin(x^2+y^2)$

11. For the function f defined by $f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$.

Verify whether $f_{xy}(0,0) = f_{yx}(0,0)$.

12. If $f(x,y) = x^3y^2 + y\sin x$, where $x = \sin 2t$ and $y = \log t$, find $\frac{df}{dt}$.

13. If $z = x\log(xy) + y^3$, where $y = \sin(x^2+1)$, find $\frac{\partial z}{\partial x}$.

14. If $z = f(u,v)$, where $u = e^x\cos y$ and $v = e^x\sin y$, then show that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2 = (u^2 + v^2)\left((\frac{\partial z}{\partial u})^2 + (\frac{\partial z}{\partial v})^2\right)$.

15. Compute $\frac{\partial^2 f}{\partial x^2}$ for f(x,y), where $x = r\cos\theta$ and $y = r\sin\theta$.

16. If $y = f(x+ct) + g(x-ct)$, then show that
$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$.

17. If $u = xf(x+y) + yg(x+y)$, then show that
$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$.

18. If $u = \tan^{-1}\left(\frac{x^2+y^2}{\sqrt{x}+\sqrt{y}}\right)$, then show that
$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{4}\sin 2u$.

19. If $u = \sin^{-1}\sqrt{\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}}$, then show that
$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12}\left(\frac{13}{12} + \frac{\tan^2 u}{12}\right)$.

20. If $f(x,y,z) = \frac{1}{\sqrt{(x^2+y^2+z^2)}}$, show that
$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$.

21. If $u = x\phi(\frac{y}{x}) + \psi(\frac{y}{x})$, then show that

   (a) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\phi(\frac{y}{x})$.
   (b) $x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 0$.

22. If $f(x,y) = x^3 y + e^{xy^2}$, find f_x and f_y.

23. If $f(x,y) = xy\frac{x^2-y^2}{x^2+y^2}$, when $x^2+y^2 \neq 0$, and $f(0,0) = 0$, show that
$f_x(x,0) = 0 = f_y(0,y)$
$f_x(0,y) = -y, f_y(x,0) = x$

24. If $f(x,y) = \begin{cases} \frac{x^2-yx}{x+y}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$, find $f_x(0,0)$ and $f_y(0,0)$.

25. If $f(x,y) = \begin{cases} \frac{x^3+y^3}{x-y}, & x \neq y, \\ 0, & x = y \end{cases}$, show that the function is discontinuous at the origin but possesses partial derivatives f_x and f_y at every point, including the origin.

26. If $f(x,y) = \begin{cases} xy\tan\frac{x}{y}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$, show that
$xf_x + yf_y = 2f$.

27. Calculate $f_x, f_y, f_x(0,0), f_y(0,0)$ for the following:

   (a) $f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2}, & x \neq 0, y \neq 0, \\ 0, & x = 0 = y. \end{cases}$
   (b) $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0, \\ 0, & x^2+y^2 = 0. \end{cases}$

28. Show that the function
$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2}, & x^2+y^2 \neq 0, \\ 0, & (x,y) = (0,0). \end{cases}$, possesses first partial derivatives everywhere, including the origin, but the function is discontinuous at the origin.

29. If $f(x,y) = \sqrt{|xy|}$, find $f_x(0,0)$ and $f_y(0,0)$.

30. Verify that $f_{xy} = f_{yx}$ for the following functions:

   (a) $\frac{2x-y}{x+y}$,
   (b) $x\tan xy$,
   (c) $\cosh(y+\cos x)$,
   (d) x^y.

   indicating possible exceptional points and investigate these points.

31. Show that $z = \log\{(x-a)^2 + (y-b)^2\}$, satisfies $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$, except at (a, b).

32. Show that $z = x \cos \frac{y}{x} + \tan \frac{y}{x}$, satisfies $x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} = 0$, except at points for which $x = 0$.

33. Prove that $f_{xy} \neq f_{yx}$ at the origin for the function: $f(x,y) = \begin{cases} x^2 \tan^{-1} \frac{y}{x} - y^2 \tan^{-1} \frac{x}{y}, & x \neq 0, y \neq 0, \\ 0, & \text{elsewhere.} \end{cases}$

34. If $f(x,y,z) = \frac{1}{\sqrt{(x^2+y^2+z^2)}}$, show that
$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$

35. Examine for the change in the order of derivation at the origin for the functions:

    (a) $f(x,y) = e^x(\cos y + x \sin y)$.
    (b) $f(x,y) = \sqrt{x^2+y^2} \sin 2\phi.$, where $f(0,0) = 0$ and $\phi = \tan^{-1} \frac{y}{x}$.
    (c) $f(x,y) = |x^2 - y^2|$.

36. Examine the equality of $f_{xy}(0,0)$ and $f_{yx}(0,0)$ for the function: $f(x,y) = (x^2+y^2) \tan^{-1} \frac{y}{x}, \quad x \neq 0, f(0,y) = \frac{\pi y^2}{2}$.

37. Given $u = e^x \cos y + e^y \sin z$, find all first partial derivatives and verify that
$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}; \frac{\partial^2 u}{\partial x \partial z} = \frac{\partial^2 u}{\partial z \partial x}; \frac{\partial^2 u}{\partial y \partial z} = \frac{\partial^2 u}{\partial z \partial y}$.

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